TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 266, Number 2, August 1981

INTEGRALGEOMETRIC PROPERTIES OF CAPACITIES

BY

PERTTI MATTILA[1]

ABSTRACT. Let m and n be positive integers, $0 < m < n$, and C_K and C_H the usual potential-theoretic capacities on R^n corresponding to lower semicontinuous kernels K and H on $R^n \times R^n$ with $H(x,y) = K(x,y)|x-y|^{n-m} \geq 1$ for $|x-y| \leq 1$. We consider relations between the capacities $C_K(E)$ and $C_H(E \cap A)$ when $E \subset R^n$ and A varies over the m-dimensional affine subspaces of R^n. For example, we prove that if E is compact, $C_K(E) \leq c \int C_H(E \cap A)\, d\lambda_{n,m}A$ where $\lambda_{n,m}$ is a rigidly invariant measure and c is a positive constant depending only on n and m.

1. Introduction. Suppose that E is a Borel set in R^n with $0 < \mathcal{H}^s(E) < \infty$, where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. It was shown in **[MP]** that if m is an integer, $0 < m < n$, then, in the case $s \leq m$, the Hausdorff dimension $\dim p(E)$ equals s for almost all orthogonal projections $p\colon R^n \to R^m$, and, in the case $s > n - m$, $\dim(E \cap V) = s + m - n$ with $\mathcal{H}^{s+m-n}(E \cap V) < \infty$ for almost all m planes V through almost all points of E. There are examples which show that in general the statements $\dim p(E) = s$ and $\dim(E \cap V) = s + m - n$ cannot be replaced by $\mathcal{H}^s(p(E)) > 0$ and $\mathcal{H}^{s+m-n}(E \cap V) > 0$, respectively; see **[M**, 5.6 and 6.6]. However, if one uses capacities in place of Hausdorff measures, one can say more. If C_s is the Riesz capacity defined via the kernel $|x-y|^{-s}$, then, in the case $0 < s < m$, $C_s(E) > 0$ implies $C_s(p(E)) > 0$ for almost all orthogonal projections $p\colon R^n \to R^m$. Moreover, if F is compact, then

$$\int C_s(p(F))^{-1}\, d\theta^*_{n,m}p \leq c C_s(F)^{-1}$$

where $\theta^*_{n,m}$ is the orthogonally invariant measure on the space of all orthogonal projections $R^n \to R^m$ and c is a constant depending on n, m and s. These results were proved in **[MP**, 5.1–2], and Kaufman also considered capacities of projections in **[K]**. In 4.11 we show that also

$$C_s(F) \leq c \int C_s(p(F))\, d\theta^*_{n,m}p.$$

The main portion of this paper is devoted to the study of the capacities of the intersections of E with m planes. We shall show that if $s > n - m$ and $C_s(E) > 0$, then $C_{s+m-n}(E \cap V) > 0$ for almost all m planes V through C_s almost all points of E, and if F is compact

$$C_s(F) \leq c \int C_{s+m-n}(F \cap V)\, d\lambda_{n,m}V,$$

Received by the editors August 1, 1980.
AMS (MOS) subject classifications (1970). Primary 28A12, 31B15; Secondary 28A75.
[1]Supported in part by National Science Foundation Grant MCS77-18723(02).

0002-9947/81/0000-0362/$05.00

where $\lambda_{n,m}$ is the rigidly invariant measure on the space of all m-dimensional affine subspaces of R^n and c depends only on n and m (see 4.6–8). In fact, we prove more general results by replacing the kernels $|x-y|^{-s}$ and $|x-y|^{-(s+m-n)}$ by general lower semicontinuous kernels $K(x, y)$ and $K(x, y)|x-y|^{n-m}$, respectively.

To prove these results we need to define the slices of a Radon measure μ on m planes and to derive some integral relations between these slices and μ. This will be done in §3.

In §5 we make some remarks on the structure of purely unrectifiable subsets of R^n. For example, it follows from Theorem 5.1 that if $E \subset R^n$ is purely $(\mathcal{H}^{n-1}, n-1)$ unrectifiable with $\mathcal{H}^{n-1}(E) < \infty$, then from C^*_{n-1} almost all points of R^n E projects radially into a set of $\mathcal{H}^{n-1}$ measure zero, where C^*_{n-1} is the outer capacity corresponding to C_{n-1}. This generalizes a result of Marstrand [**M**, §8].

2. Preliminaries.

2.1. *Notation and terminology*. We shall use the notation and terminology of [**F**]. In the whole paper m and n will be integers with $0 < m < n$. Radon measure always means a nonnegative (outer) Radon measure. If μ is a Radon measure on R^n, so are $\mu \llcorner A$: $B \mapsto \mu(A \cap B)$ for any $A \subset R^n$ and, if the support of μ, spt μ, is compact, $f_\# \mu$: $B \mapsto \mu(f^{-1}(B))$ for any continuous f: $R^n \to R^n$ [**F**, 2.2.17]. If μ is absolutely continuous with respect to ν, we denote $\mu \ll \nu$.

We let $G(n, m)$ be the Grassmannian manifold of m-dimensional linear subspaces of R^n. There is a unique Radon measure $\gamma_{n,m}$ on $G(n, m)$ which has a total mass one and which is invariant under orthogonal transformations of R^n [**F**, 2.7.16(6)]. The following lemma was proved in [**MP**, 2.6]:

2.2. Lemma. *There is a constant c depending only on n and m such that for* $x \in R^n$ *and* $\delta > 0$,

$$\gamma_{n,m}\{V: \operatorname{dist}(x, V) \leqslant \delta\} \leqslant c\delta^{n-m}|x|^{m-n}.$$

2.3. *The space of affine subspaces*. We shall denote by $A(n, m)$ the space of all m-dimensional affine subspaces of R^n. Each $A \in A(n, m)$ has a unique representation

$$A = V_a, \qquad V \in G(n, m), a \in V^\perp,$$

where $V^\perp$ is the orthogonal complement of V and $V_a = V + \{a\}$ is the m plane through a parallel to V. We let $\lambda_{n,m}$ be the standard Radon measure on $A(n, m)$ which is invariant under the isometries of R^n. It follows from [**F**, 2.7.16(7)] and Fubini's theorem [**F**, 2.6.2] that

$$\int f \, d\lambda_{n,m} = \int \int_{V^\perp} f(V_a) \, d\mathcal{H}^{n-m}a \, d\gamma_{n,m} V$$

for any nonnegative Borel function f on $A(n, m)$.

2.4. *Differentiation of measures*. We review the facts from the theory of the relative differentiation of measures which will be needed in the sequel. Let $V \in G(n, k)$ and let μ be a Radon measure on R^n. We define for $x \in V$ the lower and upper derivatives of $\mu \llcorner V$ with respect to $\mathcal{H}^k \llcorner V$ by

$$\underline{D}(\mu, V, x) = \liminf_{r\downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V),$$

$$\overline{D}(\mu, V, x) = \limsup_{r\downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V),$$

where $B(x, r)$ is the closed ball with centre x and radius r and $\alpha(k)$ is the Lebesgue measure of the unit ball in R^k. If these two limits agree, we define the derivative

$$D(\mu, V, x) = \lim_{r\downarrow 0} \alpha(k)^{-1} r^{-k} \mu(B(x, r) \cap V).$$

It follows from [**F**, 2.9.5] combined with [**F**, 2.8.7–8] that

$$D(\mu, V, x) < \infty \quad \text{for } \mathcal{H}^k \text{ a.a. } x \in V,$$

and from [**F**, 2.9.7] that for any $\mathcal{H}^k$ measurable set $B \subset V$,

$$\int_B D(\mu, V, x)\, d\mathcal{H}^k x \leqslant \mu(B).$$

The equality holds if $\mu \ll \mathcal{H}^k \llcorner V$, and then it follows $D(\mu, V, x) > 0$ for μ a.a. $x \in V$. Moreover, using [**F**, 2.9.15] one sees that $\mu \llcorner V \ll \mathcal{H}^k \llcorner V$ if and only if

$$\underline{D}(\mu, V, x) < \infty \quad \text{for } \mu \text{ a.a. } x \in V.$$

2.5. *Some Borel functions*. For $\emptyset \neq E \subset R^n$ and $\delta > 0$, we set

$$E(\delta) = \{x: \operatorname{dist}(x, E) \leqslant \delta\}.$$

Suppose that μ is a Radon measure on R^n with compact support, $W \in G(n, k)$, f is a nonnegative Borel function on R^n and α is a real number. Then the following functions are Borel functions:

$$x \mapsto \underline{D}(\mu, W, x), \quad x \in W, \qquad x \mapsto \overline{D}(\mu, W, x), \quad x \in W,$$

$$A \mapsto \liminf_{\delta\downarrow 0} \delta^\alpha \int_{A(\delta)} f\, d\mu, \qquad A \in A(n, m),$$

$$A \mapsto \limsup_{\delta\downarrow 0} \delta^\alpha \int_{A(\delta)} f\, d\mu, \qquad A \in A(n, m),$$

$$(x, V) \mapsto \liminf_{\delta\downarrow 0} \delta^\alpha \int_{V_x(\delta)} f\, d\mu, \qquad (x, V) \in R^n \times G(n, m),$$

$$(x, V) \mapsto \limsup_{\delta\downarrow 0} \delta^\alpha \int_{V_x(\delta)} f\, d\mu, \qquad (x, V) \in R^n \times G(n, m).$$

The proofs of these facts are rather standard, and we briefly consider only

$$F(A) = \liminf_{\delta\downarrow 0} \delta^\alpha \int_{A(\delta)} f\, d\mu.$$

The others can be dealt with similarly. First one verifies

$$F(A) = \liminf_{\delta\downarrow 0} \delta^\alpha \int_{A(\delta)^0} f\, d\mu$$

where $A(\delta)^0$ is the interior of $A(\delta)$. Using

$$\int_{A(\delta)^0} f\, d\mu = \lim_{\varepsilon\uparrow\delta} \int_{A(\varepsilon)^0} f\, d\mu,$$

and the compactness of spt μ, one then shows that $A \mapsto \int_{A(\delta)^0} f\,d\mu$ is lower semicontinuous. Finally it follows from the facts that δ^α is continuous and $\int_{A(\delta)^0} f\,d\mu$ is nondecreasing with respect to δ that δ may be restricted to run through the positive rationals. This implies that F is a Borel function.

3. Slicing of measures. We shall use the theory of differentiation of measures to define the slices of a Radon measure of R^n on affine subspaces of R^n. Our method is somewhat similar to those which Federer [**F**, 4.3] and Almgren [**A**, I.3] have used to slice currents and varifolds.

3.1. *Definition of slices.* Let μ be a Radon measure on R^n with compact support. For any nonnegative Borel function f on R^n with $\int f\,d\mu < \infty$ we define a Radon measure ν_f by

$$\nu_f(B) = \int_B f\,d\mu.$$

Let $V \in G(n, m)$ and let $\pi_V\colon R^n \to V^\perp$ be the orthogonal projection. First we fix $\varphi \in C^+(R^n)$, the space of nonnegative continuous functions on R^n. Using 2.4 we differentiate $\pi_{V\#}\nu_\varphi$ with respect to $\mathcal{H}^{n-m} \llcorner V^\perp$ and obtain the existence of

$$\mu_{V,a}(\varphi) = D(\pi_{V\#}\nu_\varphi, V^\perp, a) = \lim_{\delta\downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int_{V_a(\delta)} \varphi\,d\mu < \infty \tag{3.2}$$

for $\mathcal{H}^{n-m}$ a.a. $x \in V^\perp$.

Let then D be a countable subset of $C^+(R^n)$ which is dense in $C^+(R^n)$ with respect to the uniform convergence. Then for $\mathcal{H}^{n-m}$ a.a. $a \in V^\perp$, $\mu_{V,a}(\varphi)$ is defined for all $\varphi \in D$, and it follows immediately that for every such a, $\mu_{V,a}(\varphi)$ is defined by (3.2) for all $\varphi \in C^+(R^n)$. Thus for $\mathcal{H}^{n-m}$ a.a. $a \in V^\perp$ we may use Riesz's representation theorem [**F**, 2.5.13–14] to extend $\mu_{V,a}$ to a Radon measure on R^n. Whenever $\mu_{V,a}$ is defined, we set

$$\mu_{V,x} = \mu_{V,a} \text{ for } x \in V_a \quad\text{and}\quad \mu_A = \mu_{V,a} \text{ for } A = V_a \in A(n, m).$$

3.3. LEMMA. (1) spt $\mu_A \subset A \cap$ spt μ *whenever* μ_A *is defined.*

(2) *The set* P *of those* $A \in A(n, m)$ *for which* μ_A *is defined is a Borel set and* $\lambda_{n,m}(A(n, m) \sim P) = 0$.

(3) *The set* Q *of all pairs* $(x, V) \in R^n \times G(n, m)$ *for which* $\mu_{V,x}$ *is defined is a Borel set. If* $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ *for* $\gamma_{n,m}$ *a.a.* $V \in G(n, m)$, *then* $\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0$ *and* $\mu_{V,x}(R^n) > 0$ *for* $\mu \times \gamma_{n,m}$ *a.a.* $(x, V) \in R^n \times G(n, m)$.

PROOF. (1) is obvious by (3.2). To prove (2) let D be the countable dense subset of $C^+(R^n)$ which was used in 3.1. For $\varphi \in D$ the functions

$$\underline{D}_\varphi\colon A \mapsto \liminf_{\delta\downarrow 0}\ \alpha(n-m)^{-1}\delta^{m-n}\int_{A(\delta)} \varphi\,d\mu,$$

$$\overline{D}_\varphi\colon A \mapsto \limsup_{\delta\downarrow 0}\ \alpha(n-m)^{-1}\delta^{m-n}\int_{A(\delta)} \varphi\,d\mu$$

are Borel functions on $A(n, m)$ by 2.5. Hence

$$P = \bigcap_{\varphi \in D} \{A: \underline{D}_\varphi(A) = \overline{D}_\varphi(A) < \infty\}$$

is a Borel set. Since $\mathcal{H}^{n-m}\{a \in V^\perp: V_a \notin P\} = 0$ for all $V \in G(n, m)$, we obtain $\lambda_{n,m}(A(n, m) \sim P) = 0$.

To prove (3) we observe that the mapping F: $(x, V) \mapsto V_{\pi_V(x)}$ of $R^n \times G(n, m)$ onto $A(n, m)$ is continuous and $Q = F^{-1}(P)$. Thus Q is a Borel set.

Suppose then that for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$, which means that $\mu(E) = 0$ whenever $\mathcal{H}^{n-m}(\pi_V(E)) = 0$. Since $\lambda_{n,m}(A(n, m) \sim P) = 0$, we have

$$\begin{aligned}\mathcal{H}^{n-m}&\big(\pi_V\{x: V_{\pi_V(x)} \in A(n, m) \sim P\}\big) \\ &= \mathcal{H}^{n-m}\{a \in V^\perp: V_a \in A(n, m) \sim P\} = 0\end{aligned}$$

for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. Hence, by the absolute continuity,

$$\mu\{x: (x, V) \in R^n \times G(n, m) \sim Q\} = \mu\{x: V_{\pi_V(x)} \in A(n, m) \sim P\} = 0$$

for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, and Fubini's theorem yields

$$\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0.$$

Finally, if $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$, we have by 2.4, $\mu_{V,a}(R^n) = D(\pi_{V\#}\mu, V^\perp, a) > 0$ for $\pi_{V\#}\mu$ a.a. $a \in V^\perp$, and then $\mu_{V,x}(R^n) > 0$ for μ a.a. $x \in R^n$. Fubini's theorem implies $\mu_{V,x}(R^n) > 0$ for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$.

3.4. LEMMA. *Let f be a nonnegative Borel function on R^n with $\int f\,d\mu < \infty$ and let $V \in G(n, m)$.*

(1) *For $\mathcal{H}^{n-m}$ a.a. $a \in V^\perp$,*

$$\int f\,d\mu_{V,a} = \lim_{\delta \downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int_{V_a(\delta)} f\,d\mu.$$

(2) *The function $a \mapsto \int f\,d\mu_{V,a}$ is $\mathcal{H}^{n-m}$ measurable on $V^\perp$.*

(3) $\int_{V^\perp}\int f\,d\mu_{V,a}\,d\mathcal{H}^{n-m}a \leqslant \int f\,d\mu$.

(4) *If $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$, then*

$$\int_{V^\perp}\int f\,d\mu_{V,a}\,d\mathcal{H}^{n-m}a = \int f\,d\mu.$$

(5) *The function $A \mapsto \int f\,d\mu_A$ is $\lambda_{n,m}$ measurable on $A(n, m)$.*

PROOF. The set $P_V = \{a \in V^\perp: V_a \in P\}$ is a Borel set by Lemma 3.3(2) and $\mathcal{H}^{n-m}(V^\perp \sim P_V) = 0$ by 3.1.

Let g be a nonnegative lower semicontinuous function on R^n. Then there is a nondecreasing sequence (φ_i) of continuous functions with $\lim \varphi_i = g$. For $a \in P_V$ by (3.2),

$$\int g\,d\mu_{V,a} = \lim_{i\to\infty}\int \varphi_i\,d\mu_{V,a} = \lim_{i\to\infty}\lim_{\delta\downarrow 0}\alpha(n-m)^{-1}\delta^{m-n}\int_{V_a(\delta)}\varphi_i\,d\mu.$$

Therefore $a \mapsto \int g \, d\mu_{V,a}$ is a Borel function by 2.5, and using the monotone convergence theorem and 2.4 we get

$$\int_{V^\perp} \int g \, d\mu_{V,a} \, d\mathcal{H}^{n-m} a = \int_{V^\perp} \lim_{i\to\infty} D(\pi_{V\#}\nu_{\varphi_i}, V^\perp, a) \, d\mathcal{H}^{n-m} a$$

$$= \lim_{i\to\infty} \int_{V^\perp} D(\pi_{V\#}\nu_{\varphi_i}, V^\perp, a) \, d\mathcal{H}^{n-m} a \leq \lim_{i\to\infty} \pi_{V\#}\nu_{\varphi_i}(V^\perp)$$

$$= \lim_{i\to\infty} \int \varphi_i \, d\mu = \int g \, d\mu.$$

Since $\int f \, d\mu < \infty$ there are sequences (ψ_i) of continuous functions and (g_i) of lower semicontinuous functions such that

$$|f - \psi_i| \leq g_i \quad \text{and} \quad \lim_{i\to\infty} \int g_i \, d\mu = 0.$$

Then $\int_{V^\perp} \int g_i \, d\mu_{V,a} \, d\mathcal{H}^{n-m} a \leq \int g_i \, d\mu \to 0$; hence for a subsequence, which we may assume to be the whole sequence,

$$\lim_{i\to\infty} \int g_i \, d\mu_{V,a} = 0 \quad \text{for } \mathcal{H}^{n-m} \text{ a.a. } a \in V^\perp. \tag{6}$$

We also have by 2.5 and 2.4,

$$\int \lim_{\delta \downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} |f - \psi_i| \, d\mu \, d\mathcal{H}^{n-m} a$$

$$\leq \int D(\pi_{V\#}\nu_{g_i}, V^\perp, a) \, d\mathcal{H}^{n-m} a \leq \int g_i \, d\mu \to 0.$$

Hence going once more to a subsequence without changing the notation, we may assume

$$\lim_{i\to\infty} \lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} |f - \psi_i| \, d\mu = 0 \quad \text{for } \mathcal{H}^{n-m} \text{ a.a. } a \in V^\perp. \tag{7}$$

Let R be the set of those $a \in P_V$ for which (6) and (7) hold and for which

$$\lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f \, d\mu = D(\pi_{V\#}\nu_f, V^\perp, a) < \infty.$$

Then by 2.5 and 2.4, R is a Borel set, $\mathcal{H}^{n-m}(V^\perp \sim R) = 0$ and for $a \in R$ and for all i,

$$\lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} f \, d\mu = \lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} \psi_i \, d\mu$$

$$+ \lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} (f - \psi_i) \, d\mu$$

$$= \int \psi_i \, d\mu_{V,a} + \lim_{\delta\downarrow 0} \alpha(n-m)^{-1} \delta^{m-n} \int_{V_a(\delta)} (f - \psi_i) \, d\mu$$

$$\to \int f \, d\mu_{V,a} \quad \text{as } i \to \infty.$$

Hence for $a \in R$,

$$\lim_{\delta \downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int_{V_a(\delta)} f\,d\mu = \int f\,d\mu_{V,a}.$$

The left-hand side is a Borel function by 2.5, whence $a \mapsto \int f\,d\mu_{V,a}$ is $\mathcal{H}^{n-m}$ measurable. This proves (1) and (2). (3) and (4) follow from (1) and 2.4 when applied to the measure $\pi_{V\#}\nu_f$.

Essentially the same argument which was used to prove (1) gives for $\lambda_{n,m}$ a.a. $A \in A(n, m)$,

$$\int f\,d\mu_A = \lim_{\delta \downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int_{A(\delta)} f\,d\mu.$$

The only difference is that one now performs integrations also over $G(n, m)$. (5) follows then from 2.5.

The following lemma, which is a generalization of [**M**, Lemma 13], gives a sufficient condition for $\pi_{V\#}\mu$ to be absolutely continuous with respect to $\mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. An immediate corollary is the fact that if $C_{n-m}(E) > 0$ or $\dim E > n - m$ then $\mathcal{H}^{n-m}(\pi_V(E)) > 0$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$. More general results involving multiplicities of the projections were derived in [**MP**, §4].

3.5. Lemma. *If* $\int |x-y|^{m-n}\,d\mu y < \infty$ *for* μ *a.a.* $x \in R^n$, *then* $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ *for* $\gamma_{n,m}$ *a.a.* $V \in G(n, m)$.

Proof. By Fatou's lemma, Fubini's theorem and Lemma 2.2, we obtain for μ a.a. $x \in R^n$,

$$\begin{aligned}
&\int \underline{D}(\pi_{V\#}\mu, V^\perp, \pi_V(x))\,d\gamma_{n,m}V \\
&\qquad \leq \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int \mu\{y: \operatorname{dist}(y, V_x) < \delta\}\,d\gamma_{n,m}V \\
&\qquad = \liminf_{\delta \downarrow 0} \alpha(n-m)^{-1}\delta^{m-n}\int \gamma_{n,m}\{V: \operatorname{dist}(y, V_x) < \delta\}\,d\mu y \\
&\qquad \leq c\alpha(n-m)^{-1}\int |x-y|^{m-n}\,d\mu y < \infty.
\end{aligned}$$

Hence for $\gamma_{n,m}$ a.a. $V \in G(n, m)$,

$$\underline{D}(\pi_{V\#}\mu, V^\perp, \pi_V(x)) < \infty \quad \text{for } \mu \text{ } a.a. \text{ } x \in R^n,$$

and therefore by the definition of $\pi_{V\#}\mu$,

$$\underline{D}(\pi_{V\#}\mu, V^\perp, a) < \infty \quad \text{for } \pi_{V\#}\mu \text{ a.a. } a \in V^\perp,$$

which according to 2.4 means $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$.

Observe in the following lemma that the μ exceptional set is independent of the Borel function f. This will be needed later on.

3.6. LEMMA. *Suppose that* $\int |x-y|^{m-n}\,d\mu y < \infty$ *for* μ *a.a.* $x \in R^n$. *There are a constant* c *depending only on* n *and* m, *and* $B \subset R^n$ *such that* $\mu(R^n \sim B) = 0$ *and that for every nonnegative Borel function* f *on* R^n,

$$\int\int f\,d\mu_{V,x}\,d\gamma_{n,m}V \leqslant c\int f(y)|x-y|^{m-n}\,d\mu y \quad \textit{for } x \in B.$$

PROOF. We reduce the proof to the case where f is continuous by first approximating f from above with lower semicontinuous functions and then approximating these lower semicontinuous functions from below by continuous functions. When f is continuous, (3.2) holds with φ replaced by f and V_a replaced by V_x for $(x, V) \in Q$, where Q is the Borel set of 3.3(3). Let B be the set of all $x \in R^n$ for which $\gamma_{n,m}\{V\colon (x, V) \in R^n \times G(n, m) \sim Q\} = 0$. Since

$$\mu \times \gamma_{n,m}(R^n \times G(n, m) \sim Q) = 0$$

by Lemmas 3.5 and 3.3(3), Fubini's theorem gives $\mu(R^n \sim B) = 0$. For $x \in B$ we use (3.2), Fatou's Lemma, Fubini's theorem and Lemma 2.2, and let g_δ be the characteristic function of the set $\{(y, V)\colon \operatorname{dist}(y, V_x) \leqslant \delta\}$ to obtain

$$\begin{aligned}\iint f\,d\mu_{V,x}\,d\gamma_{n,m}V &\leqslant \liminf_{\delta\downarrow 0}\ \alpha(n-m)^{-1}\delta^{m-n}\int\int_{V_x(\delta)} f\,d\mu\,d\gamma_{n,m}V\\ &= \liminf_{\delta\downarrow 0}\ \alpha(n-m)^{-1}\delta^{m-n}\int\int f(y)g_\delta(y, V)\,d\mu y\,d\gamma_{n,m}V\\ &= \liminf_{\delta\downarrow 0}\ \alpha(n-m)^{-1}\delta^{m-n}\int f(y)\gamma_{n,m}\{V\colon \operatorname{dist}(y, V_x) \leqslant \delta\}\,d\mu y\\ &\leqslant \alpha(n-m)^{-1}c\int f(y)|x-y|^{m-n}\,d\mu y.\end{aligned}$$

4. Energies and capacities.

4.1. *Definitions*. In the following K will be a nonnegative lower semicontinuous function on $R^n \times R^n$ (which may have value ∞). The *K-energy* of a Radon measure μ on R^n is

$$I_K(\mu) = \int\int K(x, y)\,d\mu x\,d\mu y.$$

The *K-capacity* of a compact set $F \subset R^n$ is defined by

$$C_K(F) = \sup I_K(\mu)^{-1}$$

where the supremum is taken over all Radon measures μ on R^n with spt $\mu \subset F$ and $\mu(F) = 1$. For arbitrary $E \subset R^n$ we set

$$C_K(E) = \sup\{C_K(F)\colon F \text{ compact} \subset E\}.$$

If for some $s > 0$, $K(x, y) = |x-y|^{-s}$ for $x \neq y$ and $K(x, x) = \infty$, we denote C_s instead of C_K.

Let $P \subset A(n, m)$ and $Q \subset R^n \times G(n, m)$ be the Borel sets where μ_A and $\mu_{V,x}$ are defined, respectively (recall Lemma 3.3).

4.2. LEMMA. *Let μ be a Radon measure on R^n with compact support. Then the functions*

$$A \mapsto I_K(\mu_A), \qquad A \in P,$$
$$(x, V) \mapsto I_K(\mu_{V,x}), \qquad (x, V) \in Q,$$
$$(x, V) \mapsto \int K(x, y)\, d\mu_{V,x}y, \qquad (x, V) \in Q,$$

are Borel functions.

PROOF. By the monotone convergence theorem we reduce the proof to the case where K is continuous with compact support. It follows from the Stone-Weierstrass approximation theorem that there is a sequence (K_i) converging uniformly to K such that each K_i is a finite sum of functions of the form $(x, y) \mapsto \varphi(x)\psi(y)$ where φ and ψ are continuous functions on R^n with compact support. Since $\mu_A(R^n) < \infty$ for $A \in P$ and $\mu_{V,x}(R^n) < \infty$ for $(x, V) \in Q$, uniform convergence implies convergence for μ_A- and $\mu_{V,x}$-integrals. Hence we may assume that

$$K(x, y) = \varphi(x)\psi(y) \quad \text{for } x, y \in R^n$$

where φ and ψ are continuous. Then

$$I_K(\mu_A) = \int \varphi\, d\mu_A \int \psi\, d\mu_A \quad \text{for } A \in P,$$
$$I_K(\mu_{V,x}) = \int \varphi\, d\mu_{V,x} \int \psi\, d\mu_{V,x} \quad \text{for } (x, V) \in Q,$$

and

$$\int K(x, y)\, d\mu_{V,x}y = \varphi(x) \int \psi\, d\mu_{V,x} \quad \text{for } (x, V) \in Q,$$

and the result follows from (3.2) and 2.5.

4.3. LEMMA. *Let f be a real-valued function on $R^k \times R^1$ such that the function $x \mapsto f(x, t)$ is $\mathcal{H}^k$ measurable for $t \in R^1$ and the function $t \mapsto f(x, t)$ is nonincreasing and left continuous for $x \in R^k$. Then f is $\mathcal{H}^{k+1}$ measurable.*

PROOF. Let Q be the set of rational numbers. For $\alpha \in R^1$ and $r \in Q$, set

$$E_\alpha = \{(x, t): f(x, t) \geqslant \alpha\}, \qquad E_{\alpha,r} = \{x: f(x, r) \geqslant \alpha\}.$$

Then $E_{\alpha,r}$ is $\mathcal{H}^k$ measurable and

$$E_\alpha = \bigcap_{i=1}^{\infty} \bigcup_{r \in Q} \{(x, t): x \in E_{\alpha,r}\} \cap \{(x, t): r < t < r + 1/i\}.$$

Hence E_α is $\mathcal{H}^{k+1}$ measurable.

From now on we shall assume that for some positive constant b,

$$K(x, y) \geqslant b|x - y|^{m-n} \quad \text{for } (x, y) \in R^n \times R^n, |x - y| \leqslant 1. \tag{4.4}$$

We define another lower semicontinuous kernel H by

$$H(x, y) = K(x, y)|x - y|^{n-m} \quad \text{for } x \neq y,$$
$$H(x, x) = \liminf_{\substack{(y,z) \to (x,x) \\ y \neq z}} H(y, z).$$

We first derive an integralgeometric inequality for energy-integrals.

4.5. THEOREM. *There is a constant c depending only on n and m such that for any Radon measure μ on R^n with compact support,*

$$\int I_H(\mu_A)\, d\lambda_{n,m}A \leqslant cI_K(\mu).$$

PROOF. We may assume $I_K(\mu) < \infty$. Then by (4.4), $\int |x - y|^{m-n}\, d\mu y < \infty$ for μ a.a. $x \in R^n$.

Let P be the Borel set of Lemma 3.3(2). Then $\lambda_{n,m}(A(n, m) \sim P) = 0$ and $A \mapsto I_H(\mu_A)$ is a Borel function on P by Lemma 4.2. We shall use the formula

$$\int f\, d\nu = \int_0^\infty \nu\{x: f(x) \geqslant t\}\, dt$$

for the ν-integral of a nonnegative ν measurable function f. We denote for $0 \leqslant t < \infty$, $V \in G(n, m)$,

$$E_{V,t} = \left\{x \in R^n\colon \int H(x, y)\, d\mu_{V,x}y \geqslant t\right\}.$$

Then $E_{V,t}$ is a Borel set by Lemma 4.2. Recalling that spt $\mu_{V,a} \subset V_a$ and $\mu_{V,x} = \mu_{V,a}$ if $x \in V_a$, we have for $V_a \in P$,

$$\begin{aligned} I_H(\mu_{V,a}) &= \int_0^\infty \mu_{V,a}\left\{x\colon \int H(x, y)\, d\mu_{V,a}y \geqslant t\right\} dt \\ &= \int_0^\infty \mu_{V,a}\left\{x\colon \int H(x, y)\, d\mu_{V,x}y \geqslant t\right\} dt \\ &= \int_0^\infty \mu_{V,a}(E_{V,t})\, dt. \end{aligned}$$

For $\gamma_{n,m}$ a.a. $V \in G(n, m)$ the function $a \mapsto \mu_{V,a}(E_{V,t})$ is $\mathcal{H}^{n-m}$ measurable on $V^\perp$ for $0 \leqslant t < \infty$ by Lemma 3.4(2), and for such V we may apply Lemma 4.3 with $f(a, t) = \mu_{V,a}(E_{V,t})$. Integrating over $V^\perp$ we get by Fubini's theorem and Lemma 3.4(3),

$$\begin{aligned} \int_{V^\perp} I_H(\mu_{V,a})\, d\mathcal{H}^{n-m}a &= \int_{V^\perp} \int_0^\infty \mu_{V,a}(E_{V,t})\, dt\, d\mathcal{H}^{n-m}a \\ &= \int_0^\infty \int_{V^\perp} \mu_{V,a}(E_{V,t})\, d\mathcal{H}^{n-m}a\, dt \leqslant \int_0^\infty \mu(E_{V,t})\, dt. \end{aligned}$$

Finally we integrate over $G(n, m)$, use Fubini's theorem, which is justified because the set of all $(x, V, t) \in R^n \times G(n, m) \times R^1$ for which $\int H(x, y)\, d\mu_{V,x}y \geqslant t$ is a Borel set, and apply Lemma 3.6 to obtain

$$\int I_H(\mu_A)\, d\lambda_{n,m}A \leqslant \int \int_0^\infty \mu(E_{V,t})\, dt\, d\gamma_{n,m}V$$

$$= \int_0^\infty \int \mu(E_{V,t})\, d\gamma_{n,m}V\, dt$$

$$= \int_0^\infty \int \gamma_{n,m}\left\{V: \int H(x,y)\, d\mu_{V,x}y > t\right\} d\mu x\, dt$$

$$= \int \int_0^\infty \gamma_{n,m}\left\{V: \int H(x,y)\, d\mu_{V,x}y \geqslant t\right\} dt\, d\mu x$$

$$= \int \int \int H(x,y)\, d\mu_{V,x}y\, d\gamma_{n,m}V\, d\mu x$$

$$\leqslant c\int \int K(x,y)\, d\mu y\, d\mu x = cI_K(\mu).$$

4.6. Theorem. *There is a constant c depending only on n and m such that for any compact set $F \subset R^n$,*

$$C_K(F) \leqslant c\int C_H(F \cap A)\, d\lambda_{n,m}A.$$

Proof. The function $A \mapsto C_H(F \cap A)$ is upper semicontinuous on $A(n,m)$. To see this suppose $C_H(F \cap A_0) < \alpha$, $A_0 \in A(n,m)$. Then there is an open set G such that $F \cap A_0 \subset G$ and $C_H(G) < \alpha$ (see [**FB**, Lemma 2.3.4]). Since F is compact there is a neighborhood U of A_0 in $A(n,m)$ such that $F \cap A \subset G$ for $A \in U$. Hence $C_H(F \cap A) \leqslant C_H(G) < \alpha$ for $A \in U$.

We may assume $C_K(F) > 0$. Let $\varepsilon > 0$ and let μ be a Radon measure such that $\operatorname{spt}\mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) \leqslant C_K(F)^{-1} + \varepsilon$. Since $I_K(\mu) < \infty$ (4.4) implies $\int |x-y|^{m-n} d\mu y < \infty$ for μ a.a. $x \in R^n$. Let R be the set of all $A \in A(n,m)$ for which $\mu_A(R^n) > 0$. We define $\nu_A = \mu_A(R^n)^{-1}\mu_A$ for $A \in R$. Then $\operatorname{spt}\nu_A \subset F \cap A$ and $\nu_a(F \cap A) = 1$.

By Lemmas 3.5 and 3.4(4) we have

$$\int \mu_A(R^n)\, d\lambda_{n,m}A = \mu(R^n) = 1.$$

Since by Theorem 4.5, $I_H(\nu_A) = \mu_A(R^n)^{-2} I_H(\mu_A) < \infty$ for $\lambda_{n,m}$ a.a. $A \in R$, we get from Hölder's inequality and Theorem 4.5,

$$1 = \left(\int \mu_A(R^n)\, d\lambda_{n,m}A\right)^2 = \left(\int_R \mu_A(R^n) I_H(\nu_A)^{1/2} I_H(\nu_A)^{-1/2}\, d\lambda_{n,m}A\right)^2$$

$$\leqslant \left(\int_R \mu_A(R^n)^2 I_H(\nu_A)\, d\lambda_{n,m}A\right)\left(\int_R I_H(\nu_A)^{-1}\, d\lambda_{n,m}A\right)$$

$$= \left(\int_R I_H(\mu_A)\, d\lambda_{n,m}A\right)\left(\int_R I_H(\nu_A)^{-1}\, d\lambda_{n,m}A\right)$$

$$\leqslant cI_K(\mu)\int C_H(F \cap A)\, d\lambda_{n,m}A$$

$$\leqslant c\left(C_K(F)^{-1} + \varepsilon\right)\int C_H(F \cap A)\, d\lambda_{n,m}A.$$

Letting $\varepsilon \to 0$ we get the desired result.

4.7. REMARK. It is clear that the inequality of Theorem 4.6 holds for arbitrary subsets of R^n if the integral is replaced by the lower integral.

4.8. THEOREM. *If* $E \subset R^n$ *and* $C_K(E) > 0$, *then there is* $B \subset E$ *such that* $C_K(E \sim B) = 0$ *and for* $x \in B$,

$$C_H(E \cap V_x) > 0 \quad \text{for } \gamma_{n,m} \text{ a.a. } V \in G(n, m).$$

PROOF. Suppose this is false. Then there is a compact set $F \subset E$ such that $C_K(F) > 0$ and $\gamma_{n,m}\{V: C_H(E \cap V_x) = 0\} > 0$ for $x \in F$, and we can find a Radon measure μ such that $\operatorname{spt} \mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) < \infty$. By (4.4) and Lemmas 3.5 and 3.3(3), $\mu_{V,x}(R^n) > 0$ for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$. Since $\operatorname{spt} \mu_{V,x} \subset F \cap V_x$ and $F \subset E$, we have $I_H(\mu_{V,x}) = \infty$ whenever $C_H(E \cap V_x) = 0$ and $\mu_{V,x}(R^n) > 0$. Therefore

$$\gamma_{n,m}\{V: I_H(\mu_{V,x}) = \infty\} > 0 \quad \text{for } \mu \text{ a.a. } x \in F.$$

Letting f be the characteristic function of the Borel set $\{(x, V): I_H(\mu_{V,x}) = \infty\}$ (cf. Lemma 4.2), we obtain from Fubini's theorem,

$$0 < \int\int f \, d\gamma_{n,m} \, d\mu = \int\int f \, d\mu \, d\gamma_{n,m} = \int \mu\{x: I_H(\mu_{V,x}) = \infty\} \, d\gamma_{n,m} V.$$

Hence there is a set $G \subset G(n, m)$ such that $\gamma_{n,m}(G) > 0$ and $\mu\{x: I_H(\mu_{V,x}) = \infty\} > 0$ for $V \in G$. Since $\pi_{V\#} \mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$, this gives

$$\mathcal{H}^{n-m}\{a \in V^\perp: I_H(\mu_{V,a}) = \infty\} > 0 \quad \text{for } \gamma_{n,m} \text{ a.a. } V \in G.$$

Integrating and using Theorem 4.5 we get a contradiction:

$$\infty = \int\int_{V^\perp} I_H(\mu_{V,a}) \, d\mathcal{H}^{n-m} a \, d\gamma_{n,m} V \leqslant c I_K(\mu) < \infty.$$

In the case $K(x, y) = |x - y|^{m-n}$ we have the following

4.9. THEOREM. *If* $E \subset R^n$ *and* $C_{n-m}(E) > 0$, *then there is* $B \subset E$ *such that* $C_{n-m}(E \sim B) = 0$, *and for* $x \in B$, $E \cap V_x$ *is uncountable for* $\gamma_{n,m}$ *a.a.* $V \in G(n, m)$.

PROOF. If this is false, there is a compact set $F \subset E$ such that $C_{n-m}(F) > 0$ and

$$\gamma_{n,m}\{V: E \cap V_x \text{ is at most countable}\} > 0 \quad \text{for } x \in F, \tag{1}$$

and we can find a Radon measure μ such that $\operatorname{spt} \mu \subset F$, $\mu(F) = 1$ and $I_K(\mu) < \infty$ where $K(x, y) = |x - y|^{m-n}$. We define for $0 < r < \infty$

$H_r(x, y) = 1$, if $|x - y| < r$,
$H_r(x, y) = 0$, if $|x - y| \geqslant r$,
$H(x, y) = 1$, if $x = y$,
$H(x, y) = 0$, if $x \neq y$,
$K_r(x, y) = H_r(x, y)|x - y|^{m-n}$.

Then the functions H_r and K_r are lower semicontinuous and $H_r \downarrow H$ as $r \downarrow 0$. Whenever μ_A is defined, Lebesgue's bounded convergence theorem gives

$$\int \mu_A\{x\} \, d\mu_A x = \int\int H(x, y) \, d\mu_A y \, d\mu_A x = \lim_{r \downarrow 0} I_{H_r}(\mu_A),$$

and we obtain by Theorem 4.5 and Fatou's lemma

$$\int\int \mu_A\{x\}\, d\mu_A x\, d\lambda_{n,m}A \leqslant \liminf_{r\downarrow 0} \int I_{H_r}(\mu_A)\, d\lambda_{n,m}A$$
$$\leqslant c \liminf_{r\downarrow 0} I_{K_r}(\mu) = 0$$

because $I_K(\mu) < \infty$. Hence for $\lambda_{n,m}$ a.a. $A \in A(n, m)$, $\int \mu_A\{x\}\, d\mu_A = 0$ which means $\mu_A\{x\} = 0$ for all $x \in R^n$. Since $\pi_{V\#}\mu \ll \mathcal{H}^{n-m} \llcorner V^\perp$ for $\gamma_{n,m}$ a.a. $V \in G(n, m)$ by Lemma 3.5, it follows that for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$, $\mu_{V,x}\{y\} = 0$ for all $y \in R^n$. Using Lemma 3.3(3) we find for $\mu \times \gamma_{n,m}$ a.a. $(x, V) \in R^n \times G(n, m)$, $\mu_{V,x}(R^n) > 0$ and $\mu_{V,x}\{y\} = 0$ for $y \in R^n$, whence spt $\mu_{V,x}$ is uncountable. This contradicts spt $\mu_{V,x} \subset E \cap V_x$ and (1).

4.10. REMARKS. Suppose that $s > n - m$ and E is $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(E) < \infty$. Then one can use Theorem 4.5 and the well-known relations between Hausdorff measure and capacity to prove $\dim(E \cap V_x) \geqslant s + m - n$ for $\mathcal{H}^s \times \gamma_{n,m}$ a.a. $(x, V) \in E \times G(n, m)$. This is Lemma 6.4 of [**MP**].

I do not know whether there are general results similar to 4.5–4.8 in the opposite direction. Ohtsuka has considered product sets in [**O**].

As a rather immediate consequence of [**MP**, Lemma 5.1] and Hölder's inequality we can give an inequality analogous to 4.6 for the Riesz capacities of the orthogonal projections. Here $O^*(n, m)$ is the space of all orthogonal projections $R^n \to R^m$ and $\theta^*_{n,m}$ is the orthogonally invariant measure on $O^*(n, m)$ of total mass one (see [**F**, 1.7.4 and 2.7.16]).

4.11. THEOREM. *For $0 < s < m$ there is a constant c depending only on n, m and s such that for any compact set $F \subset R^n$,*

$$C_s(F) \leqslant c \int C_s(p(F))\, d\theta^*_{n,m}p \leqslant cC_s(F).$$

PROOF. The right-hand inequality follows from $C_s(p(F)) \leqslant C_s(F)$ (see [**L**, Theorem 2.9, p. 158]). To prove the left-hand inequality we may assume $C_s(F) > 0$. By [**MP**, 5.1],

$$\int C_s(p(F))^{-1}\, d\theta^*_{n,m}p \leqslant cC_s(F)^{-1}.$$

Hölder's inequality gives

$$1 = \int C_s(p(F))^{1/2} C_s(p(F))^{-1/2}\, d\theta^*_{n,m}p$$
$$\leqslant \left(\int C_s(p(F))\, d\theta^*_{n,m}p\right)^{1/2} \left(\int C_s(p(F))^{-1}\, d\theta^*_{n,m}p\right)^{1/2},$$

whence

$$C_s(F) \leqslant c\left(\int C_s(p(F))^{-1}\, d\theta^*_{n,m}p\right)^{-1} \leqslant c\int C_s(p(F))\, d\theta^*_{n,m}p.$$

4.12. REMARK. The method of [MP] does not seem to give a similar inequality for general kernels K. However, in some special cases it can be modified, for example if $K(x, y) = \sup\{-\log|x - y|, 0\}$.

5. On the structure of purely unrectifiable sets. A set $E \subset R^n$ is *m rectifiable* if $E = f(B)$ for some Lipschitzian map $f: B \to R^n$ where $B \subset R^m$ is bounded. E is called *purely* $(\mathcal{H}^m, m)$ *unrectifiable* if it contains no m rectifiable subset of positive $\mathcal{H}^m$ measure. If $\mathcal{H}^m(E) < \infty$ and E is purely $(\mathcal{H}^m, m)$ unrectifiable, then according to one of the basic results of geometric measure theory [F, 3.3.15] $\mathcal{H}^m(p(E)) = 0$ for $\theta^*_{n,m}$ a.a. $p \in O^*(n, m)$. If E is a Borel set this means that the integralgeometric measure [F, 2.10.5] $\mathcal{I}^m_1(E) = 0$.

In [M, §8] Marstrand considered radial projections of purely $(\mathcal{H}^1, 1)$ unrectifiable $\mathcal{H}^1$ measurable plane sets E for which $\mathcal{H}^1(E) < \infty$. He showed that if A is the set of all those points $a \in R^2$ from which the radial projection of E has positive linear measure, that is, $\gamma_{2,1}\{l: (E \sim \{a\}) \cap l_a \neq \varnothing\} > 0$, then $\dim A \leqslant 1$. He also gave an example of a set E with $\dim A = 1$. But it is not known whether $\mathcal{H}^1(A) = 0$ always or even $\mathcal{H}^1(A) < \infty$.

Here we generalize Marstrand's result to arbitrary dimensions, and we also give more precise information on the exceptional set. However, the above question remains unsolved.

The *outer s-capacity* of $E \subset R^n$ is

$$C^*_s(E) = \inf\{C_s(G): E \subset G, G \text{ is open}\}.$$

For Suslin sets E, $C^*_s(E) = C_s(E)$ [L, Theorem 2.8, p 156]. If $C^*_s(E) = 0$, then $\dim E \leqslant s$ [L, Theorem 3.13, p. 196].

5.1. THEOREM. *Let $E \subset R^n$ with $\mathcal{I}^m_1(E) = 0$ and let*

$$A = \{x \in R^n: \gamma_{n,n-m}\{V: E \cap V_x \neq \varnothing\} > 0\}.$$

Then $C^*_m(A) = 0$, hence $\dim A \leqslant m$, and A is purely $(\mathcal{H}^m, m)$ unrectifiable.

PROOF. Since $\mathcal{I}^m_1$ is Borel regular [F, 2.10.1], we may assume that E is a Borel set. We first show that then the set A and

$$B = \{(x, V) \in R^n \times G(n, n-m): E \cap V_x \neq \varnothing\}$$

are Suslin sets. The map $(y, x, V) \mapsto \pi_V(x - y)$ of $R^n \times R^n \times G(n, n-m)$ into R^n is continuous. Hence

$$C = E \times R^n \times G(n, n-m) \cap \{(y, x, V): \pi_V(x - y) = 0\}$$

is a Borel set. If $p: R^n \times R^n \times G(n, n-m) \to R^n \times G(n, n-m)$, $p(y, x, V) = (x, V)$, is the projection, then $B = p(C)$, and it follows from [F, 2.2.10] that B is a Suslin set. Then

$$A = \{x: \gamma_{n,n-m}\{V: (x, V) \in B\} > 0\}$$

is a Suslin set by [D, VI, 21].

For $F \subset A$ let

$$F_V = \{x \in F: E \cap V_x \neq \varnothing\} \quad \text{for } V \in G(n, n-m).$$

Then $\pi_V(F_V) \subset \pi_V(E)$, and $\mathcal{T}_1^m(E) = 0$ implies

$$\mathcal{H}^m(\pi_V(F_V)) = 0 \quad \text{for } \gamma_{n,n-m} \text{ a.a. } V \in G(n, n-m). \tag{1}$$

We shall show that the negation of either one of the assertions yields a Suslin set $F \subset A$ and a Radon measure μ such that

$$\mu(F) > 0 \quad \text{and} \quad \pi_{V\#}\mu \ll \mathcal{H}^m \llcorner V^\perp \quad \text{for } \gamma_{n,n-m} \text{ a.a. } V \in G(n, n-m). \tag{2}$$

This leads to a contradiction. For (1) and (2) imply $\mu(F_V) = 0$ for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$, while Fubini's theorem gives

$$\int \mu(F_V)\, d\gamma_{n,n-m}V = \int_F \gamma_{n,n-m}\{V: E \cap V_x \neq \varnothing\}\, d\mu x > 0.$$

Suppose first that $C_m^*(A) > 0$. Since A is a Suslin set also $C_m(A) > 0$. Hence there are a compact set $F \subset A$ and a Radon measure μ such that $\mu(F) > 0$ and $\int |x - y|^{-m}\, d\mu y < \infty$ for μ a.a. $x \in R^n$. Then (2) follows from Lemma 3.5.

Suppose then that A is not purely $(\mathcal{H}^m, m)$ unrectifiable. Then A contains an m rectifiable subset B with $\mathcal{H}^m(B) > 0$. By [**F**, 3.2.29] there is a C^1 submanifold M of R^n such that $\mathcal{H}^m(B \cap M) > 0$. Set $F = A \cap M$. Then F is a Suslin set with $\mathcal{H}^m(F) > 0$. Let $T_x \in G(n, m)$ be the tangent plane direction of M at $x \in M$ and let

$$J(V, x) = |\det(\pi_V | T_x)|$$

for $V \in G(n, n-m)$, $x \in M$. Then by [**F**, 3.2.20],

$$\int N(\pi_V | C, y)\, d\mathcal{H}^m y = \int_C J(V, x)\, d\mathcal{H}^m x \tag{3}$$

for any $\mathcal{H}^m$ measurable set $C \subset M$, where $N(\pi_V | C, y)$ is the number of points in the set $C \cap \pi_V^{-1}\{y\}$. Since $J(V, x) = 0$ if and only if $\dim(\pi_V(T_x)) < m$, we have for every $x \in M$, $J(V, x) > 0$ for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$. Hence by Fubini's theorem,

$$\int \mathcal{H}^m\{x \in F: J(V, x) = 0\}\, d\gamma_{n,n-m}V$$
$$= \int_F \gamma_{n,n-m}\{V: J(V, x) = 0\}\, d\mathcal{H}^m x = 0;$$

thus for $\gamma_{n,n-m}$ a.a. $V \in G(n, n-m)$, $\mathcal{H}^m\{x \in F: J(V, x) = 0\} = 0$. For every such V (3) implies $\mathcal{H}^m(\pi_V(C)) > 0$ whenever $C \subset F$ with $\mathcal{H}^m(C) > 0$. This means that F and $\mu = \mathcal{H}^m \llcorner F$ satisfy (2).

References

[**A**] F. J. Almgren, Jr., *Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints*, Mem. Amer. Math. Soc., vol. 4, no. 165, 1976.

[**D**] C. Dellacherie, *Ensembles analytiques, capacités, mesures de Hausdorff*, Springer-Verlag, Berlin and New York, 1972.

[**F**] H. Federer, *Geometric measure theory*, Springer-Verlag, Berlin and New York, 1969.

[**FB**] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215.

[**K**] R. Kaufman, *On Hausdorff dimension of projections*, Mathematika **15** (1968), 153–155.

[L] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, Berlin and New York, 1972.

[M] J. M. Marstrand, *Some fundamental geometrical properties of plane sets of fractional dimensions*, Proc. London Math. Soc. (3) **4** (1954), 257–302.

[MP] P. Mattila, *Hausdorff dimension, orthogonal projections and intersections with planes*, Ann. Acad. Sci. Fenn. Ser. AI **1** (1975), 227–244.

[O] M. Ohtsuka, *Capacité des ensembles produits*, Nagoya Math. J. **12** (1957), 95–130.

School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

Department of Mathematics, University of Helsinki, Helsinki, Finland (Current address)